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Saturation and resonance of nonlinear system under bounded noise excitation

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Abstract

The principal resonance of a 2dof nonlinear oscillator due to bounded random excitations is investigated. Equations of modulation of response amplitude and phase are derived by the method of multiple scales. Steady-state moments for the response amplitude of the system are determined through the linearized Ito differential equation. The results of theoretical analyses are verified by numerical simulations. Saturation phenomena are found in the random counterpart. Some recommendations for potential applications of this random saturation phenomenon to vibration control problems are given at the end of the paper.

1. Introduction

Saturation as a typical phenomenon in mdof nonlinear systems was first discovered by Nayfeh et al. while studying the coupling motion of pitch and roll of a ship [1]. This phenomenon usually happens in a system with quadratic nonlinearities subject to harmonic excitations, if the two natural frequencies are in the ratio 2:1. When the excitation frequency is near the higher mode natural frequency of the system, the higher mode response at first responds linearly with the increase of the excitation amplitude. However, when the higher mode response reaches a critical level, it will grow no more and all the additional input energies will overflow into the lower mode

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response. Recently, the nonlinear phenomenon of saturation has found its applications in vibration control. Several nonlinear vibration absorbers designed on the basis of saturation phenomena have shown their feasibility and efficiency in practice [2–7]. However, the vibration control problems dealt there are all for deterministic harmonic excitations.

Since random noise is often met in practice, it is important to investigate the influence of random excitations on the response of nonlinear systems. In this paper, the principal resonance of a 2dof nonlinear oscillator with quadratic nonlinearities subject to bounded random excitations is investigated. The phenomenon of saturation is found in its random response form. Some recommendations for potential applications of this phenomenon to vibration control problems are suggested at the end of the paper.

2. Formulation of problem

The present paper is concerned with a 2dof nonlinear system with quadratic nonlinearities under external bounded random excitations, the response problem of which is governed by the following equation:

$$\begin{aligned} &\ddot{u}_1 + \omega_1^2 u_1 + \varepsilon(2\mu_1 \dot{u}_1 - \alpha_1 u_1 u_2) = 0, \\ &\ddot{u}_2 + \omega_2^2 u_2 + \varepsilon(2\mu_2 \dot{u}_2 - \alpha_2 u_1^2) = \varepsilon \xi(t), \end{aligned} \tag{1}$$

where the overhead dots indicate differentiation with respect to time t, $\varepsilon \ll 1$ is a small parameter, u_1 denotes the response of a second-order controller (or absorber), ω_1 is the natural frequency of the controller, μ_1 is the damping ratio of the controller; u_2 denotes the single-mode response of the observed structure, ω_2 is the modal frequency of the structure, μ_2 is the modal damping ratio, α_1 and α_2 are positive gain constants, and $\xi(t)$ is a bounded random process governed by the following equation:

$$\xi(t) = f\cos(\Omega t + \bar{\gamma} W(t) + \phi), \tag{2}$$

where $f > 0$ is the amplitude of the random excitation, Ω is the center frequency, ϕ is a random phase, uniformly distributed within $(0, 2\pi)$, $W(t)$ is a standard Wiener process, and $\bar{\gamma} \geqslant 0$ is the noise intensity. According to Stratonovich [8], the power spectrum $S_\xi(\omega)$ of $\xi(t)$ is

$$S_\xi(\omega) = \frac{(f\bar{\gamma})^2}{2\pi}\left[\frac{1}{4(\Omega-\omega)^2 + \bar{\gamma}^4} + \frac{1}{4(\Omega+\omega)^2 + \bar{\gamma}^4}\right]. \tag{3}$$

Obviously $|\xi(t)| \leqslant f$, so $\xi(t)$ is a bounded random process. Model (2) covers the two opposite limiting cases of Eq. (3). The limiting procedure $\bar{\gamma} \to \infty$ and $f \to \infty$ leads to a uniform power spectrum of white noise. While $\bar{\gamma} \to 0$, the fluctuation spectrum $S_\xi(\omega)$ is vanishing over the entire frequency range except at two discrete frequencies $\omega = \pm\Omega$ where $S_\xi(\pm\Omega)$ goes to infinity, implying a harmonic excitation. In some other case, $S_\xi(\omega)$ may represent the Dryden and von Karman power spectrum of the air on-flow [9], hence Eq. (2) is a suitable model for random excitation. In this paper, only the case when $\bar{\gamma}$ is small is discussed.

When $\bar{\gamma} = \phi = 0$, i.e. the system is subject to a deterministic harmonic excitation, the internal resonance and saturation phenomenon, together with their applications to nonlinear vibration control have been extensively studied [2–7] recently. However, when $\bar{\gamma} \neq 0, \phi \neq 0$, i.e. the system is subject to a bounded random excitation, the related phenomena in the system and their potential applications are still under development. In this paper, the random response of system (1) is studied to find the saturation phenomenon and its potential applications in the random case.

3. Multiple scales method

The method of multiple scales [10,11] has been widely used in the analysis of deterministic systems. Rajan and Davies [12] and Nayfeh and Serhan [13] extended this method to the analysis of nonlinear systems under random external excitations. The present authors extended this method to the nonlinear systems under random parametric excitations [14,15]. In this paper, we try to investigate the response of system (1) by the multiple scale method. First, a uniformly approximate solution of Eqs. (1) is sought in the form

$$u_n(t,\varepsilon) = u_{n0}(T_0,T_1) + \varepsilon u_{n1}(T_0,T_1) + \cdots, \quad n = 1,2, \tag{4}$$

where $T_0 = t, T_1 = \varepsilon t$ are the fast and slow scales, respectively.

Throughout this paper we only discuss the first-order uniform expansion of the solution $u_{n0}(T_0,T_1)$ of Eqs. (1). By denoting $D_0 = \partial/\partial T_0, D_1 = \partial/\partial T_1$, the ordinary time derivatives can be transformed into partial derivatives as

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \varepsilon D_1 + \cdots, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots. \tag{5}$$

Substituting Eqs. (4) and (5) into Eq. (1) and comparing coefficients of ε with equal powers, one obtains the following equations:

$$\begin{aligned} D_0^2 u_{10} + \omega_1^2 u_{10} &= 0, \\ D_0^2 u_{20} + \omega_2^2 u_{20} &= 0, \end{aligned} \tag{6}$$

$$\begin{aligned} D_0^2 u_{11} + \omega_1^2 u_{11} &= -2D_0 D_1 u_{10} - 2\mu_1 D_0 u_{10} + \alpha_1 u_{10} u_{20}, \\ D_0^2 u_{21} + \omega_2^2 u_{21} &= -2D_0 D_2 u_{20} - 2\mu_2 D_0 u_{20} + \alpha_2 u_{10}^2 + \xi. \end{aligned} \tag{7}$$

The general solution of Eq. (6) can be written as

$$u_{n0}(T_0,T_1) = A_n(T_1)\exp(\mathrm{i}\omega_n T_0) + \mathrm{cc}, \quad n = 1,2, \tag{8}$$

where cc represents the complex conjugate of its preceding terms, and $A_n(T_1)$ is the slowly varying amplitude of the response. Substituting Eqs. (2) and (8) into Eq. (7), one obtains

$$\begin{aligned} D_0^2 u_{11} + \omega_1^2 u_{11} &= -2\mathrm{i}\omega_1(A_1' + \mu_1 A_1)\mathrm{e}^{\mathrm{i}\omega_1 T} + \alpha_1 A_1 A_2 \mathrm{e}^{\mathrm{i}(\omega_2+\omega_1)T_0} + \alpha_1 \bar{A}_1 A_2 \mathrm{e}^{\mathrm{i}(\omega_2-\omega_1)T_0} + \mathrm{cc}, \\ D_0^2 u_{21} + \omega_2^2 u_{21} &= -2\mathrm{i}\omega_2(A_2' + \mu_2 A_2)\mathrm{e}^{\mathrm{i}\omega_2 T} + \alpha_2 A_1^2 \mathrm{e}^{2\mathrm{i}\omega_1 T_0} + \alpha_2 A_1 \bar{A}_1 + \frac{f}{2}\mathrm{e}^{\mathrm{i}(\omega T_0 + \gamma W(T_1) + \phi)} + \mathrm{cc}, \end{aligned} \tag{9}$$

where the prime note stands for the derivative with respect to T_1, the overbar stands for the complex conjugate, and $\gamma = \bar{\gamma}/\sqrt{\varepsilon}$. For unit Wiener progress $W(t)$, with $EW(t) = 0$, and $EW^2(t) = t$, one has

$$\bar{\gamma} W(t) = \bar{\gamma}/\sqrt{\varepsilon} W(\varepsilon t) = \gamma W(T_1).$$

One can easily show that any particular solution of equations (9) contains secular terms proportional to $A_1' + \mu_1 A_1$ and $A_2' + \mu_2 A_2$, respectively, and small-divisor terms for internal resonance when $\omega_2 \approx 2\omega_1$, and for external resonance when $\Omega \approx \omega_2$. In this paper, the case of simultaneous internal resonance and external resonance is considered.

To express quantitatively the nearness of these resonances, we introduce the detuning parameters σ_1 and σ_2 according to

$$\omega_2 = 2\omega_1 + \varepsilon\sigma_1, \quad \Omega = \omega_2 + \varepsilon\sigma_2.$$

Then, put them into

$$(\omega_2 - \omega_1)T_0 = \omega_1 T_0 + \sigma_1 T_1, \quad 2\omega_1 T_0 = \omega_2 T_0 - \sigma_1 T_1, \quad \Omega T_0 = \omega_2 T + \sigma_2 T_1.$$

Using the above equations, one can transform the small-divisor terms in Eq. (9) proportional to $\exp[\pm \mathrm{i}(\omega_2 - \omega_1)T_0]$, $\exp(\pm 2\mathrm{i}\omega_1 T_0)$ and $\exp(\pm \mathrm{i}\Omega T_0)$ into the form of secular terms. Then, eliminating these secular terms yields the following equation for the A_n:

$$\begin{aligned} &2\mathrm{i}\omega_1(A_1' + \mu_1 A_1) - \alpha_1 \bar{A}_1 A_2 \mathrm{e}^{\mathrm{i}\sigma_1 T_1} = 0, \\ &2\mathrm{i}\omega_2(A_2' + \mu_2 A_2) - \alpha_2 A_1^2 \mathrm{e}^{-\mathrm{i}\sigma_1 T_1} + \frac{f}{2}\mathrm{e}^{\mathrm{i}\sigma_2 T_1 + \mathrm{i}\gamma W(T_1) + \mathrm{i}\phi} = 0. \end{aligned} \tag{10}$$

Put A_n in the polar form

$$A_n(T_1) = \tfrac{1}{2} a_n(T_1) \exp[\mathrm{i}\theta_n(T_1)], \quad n = 1, 2. \tag{11}$$

Substituting Eq. (11) into Eq. (10) and separating the real and imaginary parts of Eq. (10), one obtains

$$\begin{aligned} a_1' &= -\mu_1 a_1 + \frac{\alpha_1}{4\omega_1} a_1 a_2 \sin\gamma_1, \\ a_1\theta_1' &= -\frac{\alpha_1}{4\omega_1} a_1 a_2 \cos\gamma_1, \\ a_2' &= -\mu_2 a_2 - \frac{\alpha_2}{4\omega_2} a_1^2 \sin\gamma_1 + \frac{f}{2\omega_2}\sin\gamma_2, \\ a_2\theta_2' &= -\frac{\alpha_2}{4\omega_2} a_1^2 \cos\gamma_1 + \frac{f}{2\omega_2}\cos\gamma_2, \\ \gamma_1 &= \theta_2 - 2\theta_1 + \sigma_1 T_1, \quad \gamma_2 = \theta_2 - \sigma_2 T_1 - \gamma W(T_1) - \phi. \end{aligned} \tag{12}$$

Eq. (12) is a system of first-order equations governing the modulation of amplitude and phase. After solving a_n and θ_n, the first-order uniform expansion of the solution of Eq. (1) is given by

$$\begin{aligned} u_n &= \tfrac{1}{2} A_n(T_1) \exp(\mathrm{i}\omega_n T_0) + \mathrm{cc} + O(\varepsilon) \\ &= \tfrac{1}{2} a_n \exp[\mathrm{i}(\omega_n T_0 + \theta_1)] + \mathrm{cc} + O(\varepsilon) \\ &= a_n(\varepsilon t)\cos[\omega_n t + \theta_n(\varepsilon t)] + O(\varepsilon). \end{aligned} \tag{13}$$

4. Steady-state response

Since Eq. (12) is difficult to solve exactly, a perturbation method is used. Assuming γ is sufficiently small, we first determine the response of system (1) when $\gamma = 0$. In this case, Eq. (12) can be written as

$$\begin{aligned}
a_1' &= -\mu_1 a_1 + \frac{\alpha_1}{4\omega_1} a_1 a_2 \sin\gamma_1, \\
a_1\theta_1' &= -\frac{\alpha_1}{4\omega_1} a_1 a_2 \cos\gamma_1, \\
a_2' &= -\mu_2 a_2 - \frac{\alpha_2}{4\omega_2} a_1^2 \sin\gamma_1 + \frac{f}{2\omega_2}\sin\gamma_2, \\
a_2\theta_2' &= -\frac{\alpha_2}{4\omega_2} a_1^2 \cos\gamma_1 + \frac{f}{2\omega_2}\cos\gamma_2, \\
\gamma_1 &= \theta_2 - 2\theta_1 + \sigma_1 T_1, \quad \gamma_2 = \theta_2 - \sigma_2 T_1 - \phi.
\end{aligned} \tag{14}$$

Nontrivial steady responses $a_n = a_n^*, \gamma_n = \gamma_n^*$ correspond to the nontrivial fixed points of Eq. (14). That is, they satisfy

$$a_n' = \gamma_n' = 0, \quad \theta_1' = \tfrac{1}{2}(\sigma_1 + \sigma_2), \quad \theta_2' = \sigma_2$$

and are given by

$$\begin{aligned}
&-\mu_1 a_1 + \frac{\alpha_1}{4\omega_1} a_1 a_2 \sin\gamma_1 = 0, \\
&\frac{1}{2}(\sigma_1 + \sigma_2) a_1 = -\frac{\alpha_1}{4\omega_1} a_1 a_2 \cos\gamma_1, \\
&-\mu_2 a_2 - \frac{\alpha_2}{4\omega_2} a_1^2 \sin\gamma_1 + \frac{f}{2\omega_2}\sin\gamma_2 = 0, \\
&\sigma_2 a_2 = -\frac{\alpha_2}{4\omega_2} a_1^2 \cos\gamma_1 + \frac{f}{2\omega_2}\cos\gamma_2.
\end{aligned} \tag{15}$$

There are two possibilities. First,

$$a_1 = a_1^* = 0, \quad a_2 = a_2^* = \frac{f}{2\omega_2\sqrt{\mu_2^2 + \sigma_2^2}}, \tag{16}$$

which is the linear case. Second, $a_1 \neq 0$ and $a_2 \neq 0$ and Eq. (15) yields the following solution:

$$\begin{aligned}
a_1 &= a_1^* = 2\alpha_2^{-1/2}\left[-\Gamma_1 \pm \left(\frac{1}{4}f^2 - \Gamma_2^2\right)^{1/2}\right]^{1/2}, \\
a_2 &= a_2^* = 2\omega_1\alpha_1^{-1}[4\mu_1^2 + (\sigma_1 + \sigma_2)^2]^{1/2},
\end{aligned} \tag{17}$$

where

$$\Gamma_1 = 2\omega_1\omega_2\alpha_1^{-1}[2\mu_1\mu_2 - \sigma_2(\sigma_1 + \sigma_2)], \quad \Gamma_2 = 2\omega_1\omega_2\alpha_1^{-1}[2\mu_1\sigma_2 + \mu_2(\sigma_1 + \sigma_2)].$$

It is important to note in Eq. (17) that the steady-state value of amplitude a_2 is independent of the amplitude of the excitation. This phenomenon is called saturation in deterministic system.

Next, one can determine the condition that all roots in Eq. (17) are real. To this end, one identifies two critical values τ_1 and τ_2 as

$$\tau_1 = 2|\Gamma_1|, \quad \tau_2 = 2(\Gamma_1^2 + \Gamma_2^2)^{1/2}.$$

Clearly, τ_2 must be greater than or equal to τ_1. Then there are two possibilities: $\Gamma_1 \geqslant 0$ and $\Gamma_1 < 0$. In the former case, one real solution exists if $f \geqslant \tau_2$. Qualitative analyses show that when $f < \tau_2$, the steady-state solution should be in the form of Eq. (16) according to linear theory. On the other hand, when $f > \tau_2$, there are two possible solutions: a finite amplitude steady s olution that is stable, and a trivial steady-state solution that is unstable. Hence, when $f > \tau_2$, the steady-state solution should be in the form of Eq. (17) according to nonlinear theory. Moreover, Eqs. (16) and (17) show that irrespective of whatever the initial conditions are the motion achieves the same steady state. When $\Gamma_1 < 0$ and $f < \tau_1$ Eq. (17) has no real solutions and the response is in the form of Eq. (16). When $f > \tau_2$ Eq. (17) has one real solution and consequently the response is one of the following two possibilities: the solution in the form of Eq. (16) which is unstable according to the qualitative analysis, while the other one given by Eq. (17) is stable. When $\tau_1 < f < \tau_2$, Eq. (17) has two real solutions and the response has three branches, qualitative analysis shows that among them only the largest and the smallest ones are stable and realizable.

Next, we determine the effect of the noise, i.e. $\gamma \neq 0$, on the deterministic steady-state motion. To this end, we let the solution of Eq. (12) in the form

$$a_n = a_n^* + x_n, \;\; \gamma_n = \gamma_n^* + y_n, \;\; n = 1, 2, \tag{18}$$

where a_n^*, γ_n^* are given by Eqs. (15)–(17), and x_n, y_n are perturbation terms. Substituting Eq. (18) into Eq. (12) and neglecting the nonlinear terms of x_n, y_n, one obtains the linearization for the modulation Eq. (12) at (a_n^*, γ_n^*)

$$\begin{aligned}
x_1' &= \frac{1}{4\omega_1}\alpha_1 a_1^* \sin\gamma_1^* x_2 + \frac{1}{4\omega_1}\alpha_1 a_1^* a_2^* \cos\gamma_1^* y_1, \\
x_2' &= -\frac{1}{2\omega_2}\alpha_2 a_1^* \sin\gamma_1^* x_1 - \mu_2 x_2 - \frac{1}{4\omega_2}\alpha_2 (a_1^*)^2 \cos\gamma_1^* y_1 + \frac{f}{2\omega_2}\cos\gamma_2^* y_2, \\
y_1' &= -\frac{\alpha_2 a_1^* \cos\gamma_1^*}{2\omega_2 a_2^*} x_1 + \frac{\alpha_1 \cos\gamma_1^*}{2\omega_1} x_2 + \left[\frac{\alpha_2 (a_1^*)^2 \sin\gamma_1^*}{4\omega_2 a_2^*} - \frac{\alpha_1 a_2^* \sin\gamma_1^*}{2\omega_1}\right] y_1 - \frac{f \sin\gamma_2^*}{2\omega_1 a_2^*} y_2, \\
y_2' &= -\frac{\alpha_2 a_1^* \cos\gamma_1^*}{2\omega_2 a_2^*} x_1 + \frac{\alpha_2 (a_1^*)^2 \sin\gamma_1^*}{4\omega_2 a_2^*} y_1 - \frac{f \sin\gamma_2^*}{2\omega_2 a_2^*} y_2 - \gamma W'(T_1).
\end{aligned} \tag{19}$$

By denoting $X = (x_1, x_2, y_1, y_2)^{\mathrm{T}}$, Eq. (19) can be substituted into the following Ito equation:

$$\mathrm{d}X = AX\,\mathrm{d}T_1 + B\,\mathrm{d}W(T_1), \tag{20}$$

where A and B are the coefficient matrices

$$A=\begin{bmatrix} 0 & \frac{1}{4\omega_1}\alpha_1 a_1^* \sin\gamma_1^* & \frac{1}{4\omega_1}\alpha_1 a_1^* a_2^* \cos\gamma_1^* & 0 \\ -\frac{1}{2\omega_2}\alpha_2 a_1^* \sin\gamma_1^* & -\mu_2 & -\frac{1}{4\omega_2}\alpha_2 (a_1^*)^2 \cos\gamma_1^* & \frac{f}{2\omega_2}\cos\gamma_2^* \\ -\frac{\alpha_2 a_1^* \cos\gamma_1^*}{2\omega_2 a_2^*} & \frac{\alpha_1 \cos\gamma_1^*}{2\omega_1} & \frac{\alpha_2 (a_1^*)^2 \sin\gamma_1^*}{4\omega_2 a_2^*} - \frac{\alpha_1 a_2^* \sin\gamma_1^*}{2\omega_1} & -\frac{f\sin\gamma_2^*}{2\omega_1 a_2^*} \\ -\frac{\alpha_2 a_1^* \cos\gamma_1^*}{2\omega_2 a_2^*} & 0 & \frac{\alpha_2 (a_1^*)^2 \sin\gamma_1^*}{4\omega_2 a_2^*} & -\frac{f\sin\gamma_2^*}{2\omega_2 a_2^*} \end{bmatrix},$$

$$B=[0,0,0,-\gamma]^{\mathrm{T}}.$$

Eq. (20) is a linear Ito equation, so by using the Ito rule the steady-state moments Ex_n and Ex_n^2 can be obtained by the moment method [16]. For the steady-state moments, one has

$$\frac{\mathrm{d}Ex_n}{\mathrm{d}T_1}=\frac{\mathrm{d}Ey_n}{\mathrm{d}T_1}=0.$$

Taking expectation on both sides of Eq. (20), one obtains

$$Ex_n=Ey_n=0,\quad n=1,2. \tag{21}$$

Similarly, one has

$$\frac{\mathrm{d}Ex_1^2}{\mathrm{d}T_1}=\frac{\mathrm{d}Ex_1x_2}{\mathrm{d}T_1}=\frac{\mathrm{d}Ex_1y_1}{\mathrm{d}T_1}=\frac{\mathrm{d}Ex_1y_2}{\mathrm{d}T_1}=\frac{\mathrm{d}Ex_2^2}{\mathrm{d}T_1}=\frac{\mathrm{d}Ex_2y_1}{\mathrm{d}T_1}$$
$$=\frac{\mathrm{d}Ex_2y_2}{\mathrm{d}T_1}=\frac{\mathrm{d}Ey_1^2}{\mathrm{d}T_1}=\frac{\mathrm{d}Ey_1y_2}{\mathrm{d}T_1}=\frac{\mathrm{d}Ey_2^2}{\mathrm{d}T_1}=0.$$

Using the above equations and Eq. (20), one obtains

$$\begin{aligned} &EX^{\mathrm{T}}(AI_{1j}+I_{1j}A)X=0,\quad j=1,2,3,4,\\ &EX^{\mathrm{T}}(AI_{2j}+I_{2j}A)X=0,\quad j=2,3,4,\\ &EX^{\mathrm{T}}(AI_{3j}+I_{3j}A)X=0,\quad j=3,4,\\ &EX^{\mathrm{T}}(AI_{44}+I_{44}A)X=\gamma^2, \end{aligned} \tag{22}$$

where I_{ij} denotes a 4×4 matrix, with the (i,j) element of I_{ij} equal to 1 and the other elements equal to zero. Eq. (22) contains 10 linear equations, while the unknown parameters are the following 10 second-order steady-state moments:

$$Ex_1^2, Ex_1x_2, Ex_1y_1, Ex_1y_2, Ex_1y_2, Ex_2y_1, Ex_2y_2, Ey_1^2, Ey_1y_2, Ey_2^2.$$

Hence they can be solved by Eq. (22). Combining Eqs. (18), (21) and (22), one obtains

$$Ea_n=a_n^*,\quad Ea_n^2=(a_n^*)^2+Ex_n^2,\quad n=1,2. \tag{23}$$

5. Numerical simulations

For the method of numerical simulations, the reader is referred to Zhu [16] and Shinozuka [17,18]. Eq. (2) for $\xi(t)$ can be rewritten as follows:

$$\xi(t) = h\cos(\psi(t)),$$
$$\dot{\psi}(t) = \Omega + \gamma\zeta(t), \quad \zeta(t) = \dot{W}(t), \tag{24}$$

where Gaussian white noise $\zeta(t)$ stands for the formal derivative of a standard Wiener process. Since white noise has a uniform power spectrum and is physically unrealizable, for numerical simulations one may take a broad-band one for instead, e.g. take the power spectrum of $\zeta(t)$ as

$$S_\zeta(\omega) = \begin{cases} 1, & 0<\omega\leqslant 2\Omega, \\ 0, & \omega>2\Omega. \end{cases} \tag{25}$$

In practice, it is more convenient to take the pseudorandom signal for $\zeta(t)$ given by [16]

$$\zeta(t) = \sqrt{\frac{4\Omega}{N}}\sum_{k=1}^{N}\cos\left[\frac{\Omega}{N}(2k-1)t + \varphi_k\right], \tag{26}$$

where φ_k's are mutually independent and uniformly distributed in $(0, 2\pi]$, and N is a large integer number.

In the following numerical simulations, the parameters for system (1) are chosen as follows:

$$\alpha_1 = \alpha_2 = 1.0, \quad \mu_1 = \mu_2 = 0.2, \quad \omega_1 = 1.0, \quad \varepsilon = 0.1.$$

The governing Eq. (1) is numerically integrated by the fourth-order Runge–Kutta algorithm, and the numerical results are shown in Figs. 1 and 2.

When $\bar{\gamma} = 0$, the variations of the steady-state response with f are shown in Fig. 1, and the theoretical results given by Eqs. (16) and (17) are also shown there for comparison.

Fig. 1 shows the response curves for a representative case: $\sigma_1 = 1.0, \sigma_2 = 0.0, \Gamma_1>0$. Both Fig. 1 and Eq. (17) clearly show a saturation phenomenon. The steady-state value of the amplitude a_2 of the directly excited mode is independent of the amplitude of the excitation f as long as it is above the critical value τ_2. However, the steady-state value of the amplitude a_1 of the indirectly excited mode (through internal resonance) increases with increasing value of the excitation amplitude.

Next, we determine the effect of the noise term $\bar{\gamma}W(t)$ on the primary response. When $\sigma_1 = 1.0, \sigma_2 = 0.0, \gamma = 0.002$, the variations of the steady-state response with f are shown in Fig. 2; for comparison, the theoretical results given by Eq. (23) are also shown in Fig. 2.

Fig. 2 shows that when γ is small enough, saturation phenomenon still exists in random excited case. The steady-state moment Ea_2^2 of the directly excited mode is independent of the amplitude of the excitation f after f reaches some critical value. One may call it random saturation. From Eq. (23), one can see the steady-state moment Ea_2 also has the phenomenon of random saturation.

One can also see the evolution of energy distribution between responses of the controller and the structure through the numerical simulation for sample time histories of $u_1(t)$ and $u_2(t)$. For $f = 3.0, \sigma_1 = 1.0, \sigma_2 = 0.0, \gamma = 0.002$, a pair of sample time histories of $u_1(t)$ and $u_2(t)$ are shown in Fig. 3.

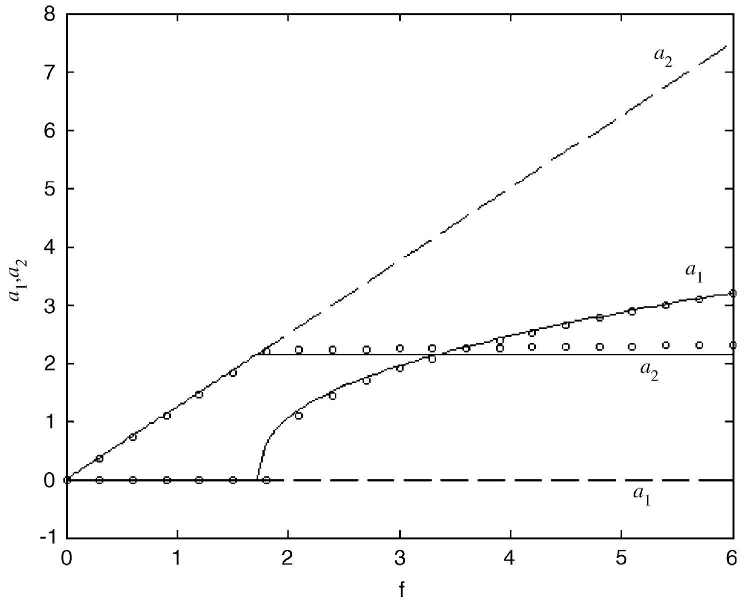

Fig. 1. Response of system (1) ($\sigma_1 = 1.0, \sigma_2 = 0.0, \bar{\gamma} = 0$): — stable solution; – – – unstable solution; ○ ○ ○ numerical solution.

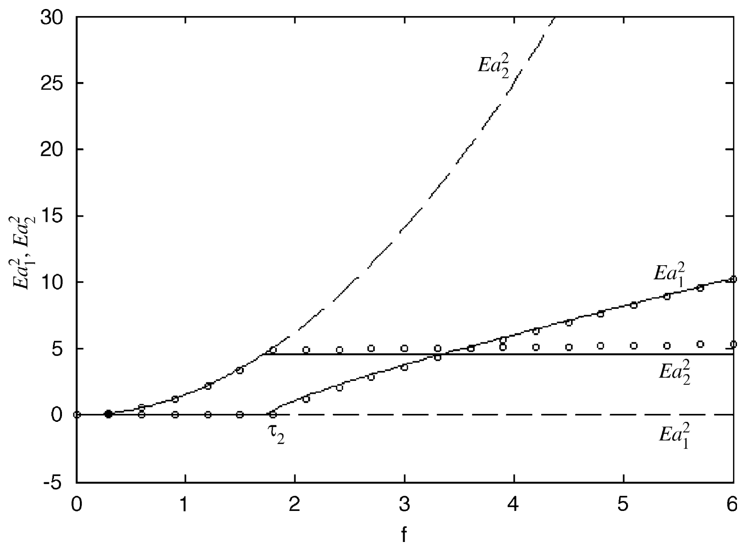

Fig. 2. Response of system (1) ($\sigma_1 = 1.0, \sigma_2 = 0.0, \gamma = 0.002$): — stable solution; – – – unstable solution; ○ ○ ○ numerical solution.

Fig. 3 shows the process of a typical control by using the random saturation and internal resonance. The energy put forward to the structure u_2 by the random excitation is largely transferred to the controller u_1. The nonlinear terms $\alpha_1 u_1 u_2$ and $\alpha_2 u_2^2$ in system (1) act as energy

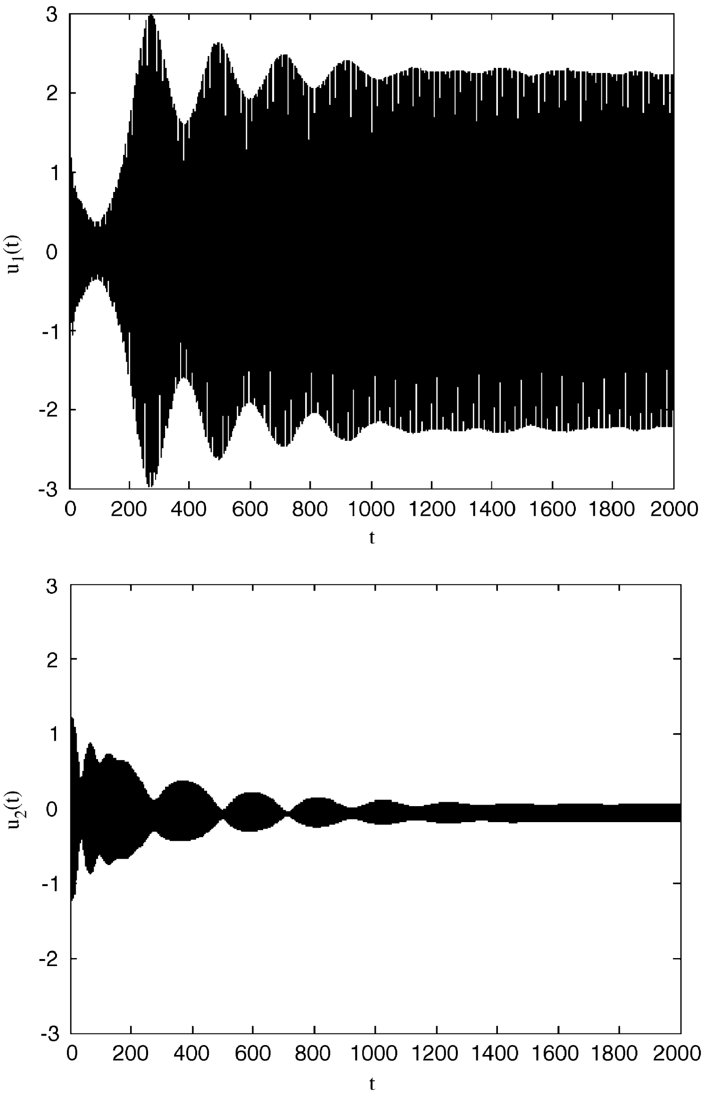

Fig. 3. Time history of $u_1(t)$ and $u_2(t)$.

bypasses to exchange energy between the structure and the controller, resulting in a somewhat beating phenomenon in response to the coupled system.

6. Conclusion and discussion

Since in system (1) the controller plays the role of a vibration absorber, where the parameters, including the natural angular frequency ω_1, the damping ratio μ_1, and the positive gain constants α_1 and α_2 are adjustable. Since u_2 represents the structural modal response to be suppressed, the natural angular frequency ω_2 and the damping ratio μ_2 of the structure are not adjustable. The amplitude f, the center frequency Ω of the random excitation, and the noise intensity $\bar{\gamma}$ are the given parameters depending upon the external environment. Hence, one could only reasonably adjust the system parameters $\alpha_1, \alpha_2, \mu_1, \sigma_1$ to bring the saturation phenomenon and resonance effect into full play to suppress the response of the structure u_2 or Ea_2^2.

The 1:2 internal resonance ($\omega_2 \approx 2\omega_1$) and the random saturation phenomenon have been used to design nonlinear controller in system (1). However, other types of vibration absorbers which have been used in the deterministic harmonic excitation case [5] can also be used in the random excitation case. First, we can use 1:3 ($\omega_2 \approx 3\omega_1$) internal resonance to design the following vibration absorber:

$$\ddot{u}_1 + \omega_1^2 u_1 + \varepsilon(2\mu_1\dot{u}_1 - \alpha_1 u_1^2 u_2) = 0,$$
$$\ddot{u}_2 + \omega_2^2 u_2 + \varepsilon(2\mu_2\dot{u}_2 - \alpha_2 u_1^3) = \varepsilon\xi(t)$$

and

$$\ddot{u}_1 + \omega_1^2 u_1 + \varepsilon(2\mu_1\dot{u}_1 - \alpha_1 u_1\dot{u}_1\dot{u}_2) = 0,$$
$$\ddot{u}_2 + \omega_2^2 u_2 + \varepsilon(2\mu_2\dot{u}_2 - \alpha_2 u_1^3) = \varepsilon\xi(t).$$

Then, we can use 1:4 ($\omega_2 \approx 4\omega_1$) internal resonance to design the following vibration absorber:

$$\ddot{u}_1 + \omega_1^2 u_1 + \varepsilon(2\mu_1\dot{u}_1 - \alpha_1 u_1^2\dot{u}_1\dot{u}_2) = 0,$$
$$\ddot{u}_2 + \omega_2^2 u_2 + \varepsilon(2\mu_2\dot{u}_2 - \alpha_2 u_1^4) = \varepsilon\xi(t)$$

and

$$\ddot{u}_1 + \omega_1^2 u_1 + \varepsilon(2\mu_1\dot{u}_1 - \alpha_1 u_1^2\dot{u}_1\dot{u}_2) = 0,$$
$$\ddot{u}_2 + \omega_2^2 u_2 + \varepsilon(2\mu_2\dot{u}_2 - \alpha_2 u_1^2\dot{u}_1^2) = \varepsilon\xi(t),$$

where u_1 represents the controller and u_2 represents the structure.

Furthermore, we can use 1:2:4 ($\omega_3 \approx 2\omega_2 \approx 4\omega_1$) internal resonance to design the following vibration absorber:

$$\ddot{u}_1 + \omega_1^2 u_1 + \varepsilon(2\mu_1\dot{u}_1 - \alpha_1 u_1\dot{u}_1\dot{u}_3) = 0,$$
$$\ddot{u}_2 + \omega_2^2 u_2 + \varepsilon(2\mu_2\dot{u}_2 - \alpha_2\dot{u}_2\dot{u}_3) = 0,$$
$$\ddot{u}_2 + \omega_3^2 u_3 + \varepsilon(2\mu_3\dot{u}_3 - \alpha_3 u_1^2\dot{u}_1^2 - \alpha_4 u_2^2) = \varepsilon\xi(t),$$

where u_1 and u_2 represent the controller and u_3 represents the structure.

Of course, the characteristics of these vibration absorbers need further research.

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